Vector Riemann-Hilbert problem with almost periodic and meromorphic coefficients and applications

By Y.A. Antipov

Department of Mathematics, Louisiana State University
Baton Rouge LA 70803, USA

Abstract

The vector Riemann-Hilbert problem is analyzed when the entries of its matrix coefficient are meromorphic and almost periodic functions. Three cases for the meromorphic functions, when they have (i) a finite number of poles and zeros (rational functions), (ii) periodic poles and zeros, and (iii) an infinite number of non periodic zeros and poles, are considered. The first case is illustrated by the heat equation for a composite rod with a finite number of discontinuities and a system of convolution equations; both problems are solved explicitly. In the second case, a Wiener-Hopf factorization is found in terms of the hypergeometric functions, and the exact solution of a mixed boundary value problem for the Laplace equation in wedge is derived. In the last case, the Riemann-Hilbert problem reduces to an infinite system of linear algebraic equations with the exponential rate of convergence. As an example, the Neumann boundary value problem for the Helmholtz equation in a strip with a slit is analyzed.

1 Introduction

Many physical models described by boundary value problems for elliptic, hyperbolic, and parabolic equations reduce to a system of n convolution equations on a finite segment. Such systems are equivalent to a vector Riemann-Hilbert problem (RHP) with a block-triangular matrix coefficient

$$G(\alpha) = \begin{pmatrix} e^{i\alpha}I & 0 \\ g(\alpha) & e^{-i\alpha}I \end{pmatrix}, \quad \alpha \in (-\infty, +\infty), \tag{1.1}$$

where I is the order n unit matrix and $g(\alpha)$ is an $n \times n$ matrix. The functions $e^{i\alpha}$ and $e^{-i\alpha}$ are almost periodic functions (Levitan, 1953), and their indices are infinite: $\operatorname{ind} e^{i\alpha} = +\infty$ and $\operatorname{ind} e^{-i\alpha} = -\infty$. If $n = 1$ and $g(\alpha)$ is a rational function, then the RHP admits a closed-form solution (Ganin, 1963). In the general case of the function $g(\alpha)$, even when $n = 1$, there is no procedure for solving the RHP with the coefficient (1.1) in closed form. Novokshenov (1980) analyzed the singular convolution equation $\int_0^a [(x-t)^{-1} + k(x-t)]u(t)dt = f(x)$, $0 < x < a$, $k(x) \in L_2(-\infty, +\infty)$, and showed that its solvability and the representation formulas for the solution are obtained in terms of the Wiener-Hopf factors of the associated matrix (1.1) ($n = 1$). A theory of factorization of matrices (1.1) when $g(\alpha)$ is an almost periodic function (including an almost periodic polynomial) was developed by Karlovich and Spitkovsky (1983), Spitkovsky (1989) and their coauthors (see for example Böttcher et al, 2002).

In many applications to physical models, $g(\alpha)$ is a meromorphic function. Antipov (1987, 1989) considered contact problems on an annular stamp and reduced it to a RHP

with the matrix coefficient (1.1), $g(\alpha) = \frac{1}{2}\Gamma(\frac{\alpha}{2})\Gamma(\gamma+\frac{1}{2}-\frac{\alpha}{2})[\Gamma(\frac{1}{2}+\frac{\alpha}{2})\Gamma(\gamma+1-\frac{\alpha}{2})]^{-1}$, $\gamma = 0, 1$. The problem was transformed into an infinite system of linear algebraic equations of the second kind with the exponential rate of convergence and solved in terms of recurrent relations. Recently, this technique for employed for the solution of an integro-differential convolution equation arising in fracture with surface effects (Antipov and Schiavone, 2011).

Another class of matrices of the form

$$G(\alpha) = \begin{pmatrix} g_{11}(\alpha) & e^{i\alpha}g_{12}(\alpha) \\ e^{-i\alpha}g_{21}(\alpha) & g_{22}(\alpha) \end{pmatrix}, \quad \alpha \in (-\infty, +\infty), \tag{1.2}$$

needs to be factorized to solve the systems of convolution equations $k_{j1} * \phi_1 + k_{j2} * \phi_2 = f_j(x)$, $a_j < x < \infty$, $\mathrm{supp}\, \phi_j \subset [a_j, \infty)$, $j = 1, 2$, $a_1 = 0$, $a_2 = a$. A method of integral equations for factorizing matrices of such a structure was proposed by Abrahams & Wickham (1990). Onishchuk (1988), Antipov & Arutyunyan (1992), Antipov (1995, 2000) worked out a technique for vector RHPs when $g_{ij}(\alpha)$ are meromorphic functions. and applied it to static fracture and contact problems. This method ultimately requires solving infinite systems of linear algebraic equations of the second kind with the exponential rate of convergence.

In this paper we aim to develop further the methodology for the RHPs whose matrix coefficient entries are meromorphic and almost-periodic functions and apply it for the solution of some model physical problems. In Section 2, we consider the heat equation for an infinite rod $u_t = a^2(x)u_{xx} + g(x,t)$ with a piece-wise constant diffusivity $a(x)$. Deconinck et al (2014) applied the Fokas method (2008) to find an exact solution in the homogeneous case, $g \equiv 0$, when the diffusivity is a piece-wise constant function, and (i) the rod is finite and the diffusivity has one or two points of discontinuity, and (ii) the rod is infinite and the function $a(x)$ is discontinuous either at 0 and ∞, or at two finite points and infinity. In particular, for an infinite rod with $a(x) = a_\pm$, $\pm x > 0$, and $g \equiv 0$, they managed to represent the solution by double integrals with one-sided Fourier internal integrals and contour external integrals over the boundary of infinite wedge-like domains. In our case, the rod is infinite, and the diffusivity has $n+1$ points of discontinuity. We show how the problem can be reduced to an order-n RHP and solved exactly in terms of the solution of an associated finite system of linear algebraic equations. In particular, when $n = 1$, and the points of discontinuity are 0 and ∞, we simplify the solution and derive an exact formula for the temperature. If $g \equiv 0$, then the solution is given by a sum of two one-dimensional integrals over the intervals $(-\infty, 0)$ and $(0, \infty)$. The formula found generalizes the classical Poisson formula for the homogeneous infinite rod. We also briefly describe another approach for the solution of the heat problem that employs the Laplace transform and the theory of discontinuous one-dimensional boundary value problems. The solution constructed by this alternative approach coincides with the one found by the RHP technique and is applicable to both cases, finite and infinite, and for any finite number of discontinuities. At the end of Section 2, we present an alternative approach for solving the Abrahams-Wickham system of convolution equations (Abrahams & Wickham, 1990) by reducing it to the case (1.2) with g_{ij} being rational functions. Notice that their procedure for the RHP hinges on a solution of an auxiliary system of integral equations, while our approach bypasses extra integral equations.

In Section 3, we propose a factorization method for matrices (1.2) when the functions g_{ij} are meromorphic and their zeros and poles are periodic. The method is illustrated by solving a mixed boundary value problem for the Laplace equation in a wedge. This

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problem for a particular choice of the boundary data was analyzed by Abrahams & Wickham (1990). They rewrote the problem as a vector RHP with the coefficient of the form (1.2) and reduced it to a system of two integral equations to be solved numerically by an approximate method. The approach we propose does not require solving any auxiliary integral equations. It derives Wiener-Hopf factors of the matrix coefficient in terms of the hypergeometric functions and ultimately yields a closed-form solution of the physical problem.

In Section 4, we generalize the method for RHPs with the matrix coefficient of the form (1.1) for the dynamic case. As an illustrative example, we take the Nemann boundary value problem for the Helmholtz equation in a strip with a finite slit. This problem describes antiplane strain deformation of a strip when the strip boundary is free of traction, and the Mode-III crack faces are subjected to oscillating loading. At the same time it could be interpreted as a model of sound transmission in a waveguide when an acoustically hard finite screen is placed inside the waveguide. We derive the solution by quadratures and some exponentially convergent series with the coefficients determined from a rapidly convergent infinite system of linear equations of the second kind.

2 Matrices with almost periodic and rational entries

In this section we derive an order-n vector RHP associated with the heat equation in a rod with a piece-wise constant diffusivity and conductivity and show that it admits a closed-form solution. To verify the procedure, we derive the solution by the standard technique of discontinuous one-dimensional boundary value problems that bypasses the RHP. To give an extra example, we solve a system of two convolution equations.

2.1 Heat equation with piece-wise constant coefficients: RHP for an order-n vector-function

The problem under consideration is one of heat conduction for an infinite rod with a piece-wise constant diffusivity $a^2(x) = k(x)/[c_p\rho(x)]$

$$u_t = a^2(x)u_{xx} + g(x,t), \quad |x| < \infty, \quad x \neq b_0, b_1, \ldots, b_{n-1}, \quad t > 0.$$

$$u|_{x=b_j^-} = u|_{x=b_j^+}, \quad k_j u_x|_{x=b_j^-} = k_{j+1}u_x|_{x=b_j^+}, \quad j = 0, 1, \ldots, n-1, \quad t \geq 0,$$

$$u|_{t=0} = f(x), \quad |x| < \infty, \tag{2.1}$$

where $u(x,t)$ is the temperature, $g(x,t) = (c_p\rho)^{-1}g_0(x,t)$, $g_0(x,t)$ is the heat source density, $f(x)$ is an initial temperature, $k(x)$ is the thermal conductivity, $\rho(x)$ is the density, c_p is the specific heat capacity, and

$$a(x) = a_j > 0, \quad k(x) = k_j, \quad \rho(x) = \rho_j, \quad x \in (b_{j-1}, b_j), \quad j = 0, 1, \ldots, n. \tag{2.2}$$

Here, we assumed $b_{-1} = -\infty$ and $b_n = +\infty$. In what follows we reduce this physical problem to an order-n vector RHP. Introduce first the Laplace transforms

$$\hat{u}(x; p) = \int_0^\infty u e^{-pt} dt, \quad \hat{g}(p) = \int_0^\infty g e^{-pt} dt, \quad \operatorname{Re} p > 0, \tag{2.3}$$

and obtain from (2.1) a discontinuous one-dimensional boundary value problem. It reads

$$a^2(x)\hat{u}_{xx} - p\hat{u} = -f - \hat{g}, \quad |x| < \infty, \quad x \neq b_0, b_1, \ldots, b_{n-1},$$

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$$\hat{u}|_{x=b_j^-} = \hat{u}|_{x=b_j^+}, \quad k_j \hat{u}_x|_{x=b_j^-} = k_{j+1} \hat{u}_x|_{x=b_j^+}, \quad j = 0, 1, \ldots, n-1. \qquad (2.4)$$

To apply further the two-sided Laplace transform, we introduce new functions of the parameter p

$$\beta_{0j}(p) = a_j^2 \hat{u}|_{x=b_j^-} - a_{j+1}^2 \hat{u}|_{x=b_j^+},$$

$$\beta_{1j}(p) = a_j^2 \hat{u}_x|_{x=b_j^-} - a_{j+1}^2 \hat{u}_x|_{x=b_j^+}, \quad j = 0, 1, \ldots, n-1, \qquad (2.5)$$

and integrate by parts

$$\int_{-\infty}^{\infty} a^2(x) \hat{u}_{xx} e^{-sx} dx = \sum_{j=0}^{n-1} (\beta_{1j} + s\beta_{0j}) e^{-sb_j} + s^2 \int_{-\infty}^{\infty} a^2(x) \hat{u} e^{-sx} dx. \qquad (2.6)$$

We split now the integral in (2.6) into $n+1$ parts and denote

$$\int_{-\infty}^{b_0} \hat{u}(x;p) e^{-sx} dx = e^{-sb_0} U_0^+(s),$$

$$\int_{b_{j-1}}^{b_j} \hat{u}(x;p) e^{-sx} dx = e^{-sb_j} U_j^+(s) = e^{-sb_{j-1}} U_j^-(s), \quad j = 1, 2, \ldots, n-1,$$

$$\int_{b_{n-1}}^{\infty} \hat{u}(x;p) e^{-sx} dx = e^{-sb_{n-1}} U_0^-(s), \qquad (2.7)$$

where

$$U_0^+(s) = \int_{-\infty}^{0} \hat{u}(x+b_0;p) e^{-sx} dx, \quad U_0^-(s) = \int_{0}^{\infty} \hat{u}(x+b_{n-1};p) e^{-sx} dx,$$

$$U_j^+(s) = \int_{b_{j-1}-b_j}^{0} \hat{u}(x+b_j;p) e^{-sx} dx, \quad U_j^-(s) = \int_{0}^{b_j-b_{j-1}} \hat{u}(x+b_{j-1};p) e^{-sx} dx,$$

$$j = 1, 2, \ldots, n-1. \qquad (2.8)$$

The functions $U_j^+(s)$ and $U_j^-(s)$ $(j = 0, 1, \ldots, n-1)$ are analytic in the domains $D^+ = \{\mathrm{Re}\, s < 0\}$ and $D^- = \{\mathrm{Re}\, s > 0\}$, respectively. We emphasize that except for $U_0^{\pm}(s)$ all the other functions $U_j^{\pm}(s)$ $(j = 1, 2, \ldots, n-1)$ are entire functions in the half-planes $D^{\mp}$. In these notations, the one-dimensional boundary value problem (2.4) can be recast as the following order-n vector RHP:

$$U_j^+(s) = e^{s(b_j-b_{j-1})} U_j^-(s), \quad j = 1, 2, \ldots, n-1,$$

$$\sum_{j=0}^{n-1} m_j(s) e^{-sb_j} U_j^+(s) + m_n(s) e^{-sb_{n-1}} U_0^-(s) = H(s), \quad s \in L, \qquad (2.9)$$

where L is the positively oriented imaginary axis (the domain D^+ is on the left), $m_j(s) = a_j^2 s^2 - p$, $j = 0, 1, \ldots, n$, and $H(s)$ is given by

$$H(s) = -\int_{-\infty}^{\infty} [f(x) + \hat{g}(x;p)] e^{-sx} dx - \sum_{j=0}^{n-1} e^{-sb_j} (\beta_{1j} + s\beta_{0j}). \qquad (2.10)$$

The vector RHP (2.9) can be transformed into a finite system of linear algebraic equations. To do this, without loss of generality, we assume $b_0 = 0$ and rearrange the RHP (2.9) as follows:

$$U_0^+(s) + \frac{m_1}{m_0} U_1^-(s) + \ldots + e^{-sb_{j-1}} \frac{m_j}{m_0} U_j^-(s) + \ldots + e^{-sb_{n-2}} \frac{m_{n-1}}{m_0} U_{n-1}^-(s)$$

$$+ e^{-sb_{n-1}} \frac{m_n}{m_0} U_0^-(s) = \frac{H(s)}{m_0},$$

$$e^{sb_j} \frac{m_0}{m_j} U_0^+(s) + \ldots + e^{s(b_j - b_{j-1})} \frac{m_{j-1}}{m_j} U_{j-1}^+(s) + U_j^+(s) + \frac{m_{j+1}}{m_j} U_{j+1}^-(s)$$

$$+ \ldots + e^{s(b_j - b_{n-1})} \frac{m_n}{m_j} U_0^-(s) = \frac{e^{sb_j} H(s)}{m_j}, \quad j = 1, 2, \ldots, n-2,$$

$$e^{sb_{n-1}} \frac{m_0}{m_{n-1}} U_0^+(s) + \ldots + e^{s(b_{n-1} - b_j)} \frac{m_j}{m_{n-1}} U_j^+(s) + \ldots + e^{s(b_{n-1} - b_{n-2})} \frac{m_{n-2}}{m_{n-1}} U_{n-2}^+(s)$$

$$+ U_{n-1}^+(s) + \frac{m_n}{m_{n-1}} U_0^-(s) = \frac{e^{sb_{n-1}} H(s)}{m_{n-1}}, \quad s \in L. \tag{2.11}$$

Notice that the functions $e^{s(b_j - b_l)} U_l^+(s)$ $(l = 0, 1, \ldots, j-1)$ are analytic in D^+ and decay exponentially as $s \to \infty$ in D^+, while the functions $e^{s(b_j - b_l)} U_{l+1}^-(s)$, $l = j+1, \ldots, n-1$ $(U_n^-(s) = U_0^-(s))$ are analytic in D^- and decay exponentially as $s \to \infty$, $s \in D^-$. On factorizing the functions

$$\frac{m_{j+1}(s)}{m_j(s)} = \frac{K_j^+(s)}{K_j^-(s)}, \quad j = 0, 1, \ldots, n-1, \tag{2.12}$$

where

$$K_j^+(s) = \frac{a_{j+1} s - \sqrt{p}}{a_j s - \sqrt{p}}, \quad K_j^-(s) = \frac{a_j s + \sqrt{p}}{a_{j+1} s + \sqrt{p}}, \quad \mathrm{Re}\sqrt{p} > 0, \tag{2.13}$$

and substituting this into equations (2.11) we have

$$\frac{U_j^+(s)}{K_j^+(s)} + \frac{1}{K_j^+(s)} \left[e^{sb_j} \frac{m_0}{m_j} U_0^+(s) + \ldots + e^{s(b_j - b_{j-1})} \frac{m_{j-1}}{m_j} U_{j-1}^+(s) \right] - \mathcal{H}_j^+(s)$$

$$= -\frac{U_{j+1}^-(s)}{K_j^-(s)} - \frac{1}{K_j^+(s)} \left[e^{s(b_j - b_{j+1})} \frac{m_{j+2}}{m_j} U_{j+2}^-(s) + \ldots + e^{s(b_j - b_{n-1})} \frac{m_n}{m_j} U_0^-(s) \right] - \mathcal{H}_j^-(s),$$

$$j = 0, 1, \ldots, n-1. \tag{2.14}$$

Here, $\mathcal{H}_j^+(s)$ and $\mathcal{H}_j^-(s)$ provide the splitting of the functions $e^{sb_j} H(s)[m_j(s) K_j^+(s)]^{-1}$ into analytic parts in the domains D^+ and D^-, respectively. In general, they are defined by the Sokhotski-Plemelj formulas

$$\mathcal{H}^{\pm}(s) = \pm \frac{e^{sb_j} H(s)}{2 m_j(s) K_j^+(s)} + \frac{1}{2\pi i} \int_L \frac{e^{\sigma b_j} H(\sigma) d\sigma}{m_j(\sigma) K_j^+(\sigma)(\sigma - s)}, \quad s \in L, \tag{2.15}$$

and the Cauchy integral is explicitly evaluated by the theory of residues. Alternatively, this splitting can be obtained by representing the function $H(s)$ as a sum of n integrals similar to (2.8) and then removing the poles. The second approach will be employed in the scalar case in section 2.2.

Now, since the poles of the left- and right-hand sides are known, we apply the Liouville theorem. Crucial to the success of the method is the fact that the known functions in the system (2.14) are meromorphic functions having a finite number of poles. That is why the left- and right-hand sides are rational functions with prescribed poles and unknown coefficients. These coefficients can be fixed by the requirement that the final solution $U_j^{\pm}(s)$ of the vector RHP has to have removable singular points at the poles lying in the half-planes $D^{\pm}$. These conditions form a finite system of linear algebraic

equations for the unknown coefficients. The case of an order-2 RHP when the functions are meromorphic and have an infinite number of periodic and not periodic poles will be considered in sections 3 and 4, respectively. The functions $\beta_{0j}(p)$ and $\beta_{0j}(p)$ introduced in (2.5) are fixes by the conditions (2.4).

2.2 Generalization of the Poisson formula for an infinite piece-wise homogeneous rod

To clarify the procedure of finding the functions $\beta_{0j}(p)$ and $\beta_{1j}(p)$, we consider the heat equation for an infinite rod composed of two semi-infinite rods having different constant diffusivities and conductivities. We assume that the initial temperature, $f_0(x)$, does not vanish at $\pm\infty$ that is

$$f_0(x) = \gamma_- \theta(-x) + \gamma_+ \theta(x) + f(x), \quad f(x) \in L_1(-\infty, \infty), \tag{2.16}$$

where $\gamma_\pm$ are nonzero constants and $\theta(x) = 1$, $x > 0$ and vanishes otherwise. It is convenient to split the temperature, $u_0(x, t)$ ($|x| < \infty$, $t \geq 0$), as

$$u_0(x, t) = u(x, t) + \gamma_- \theta(-x) + \gamma_+ \theta(x) \tag{2.17}$$

and determine $u(x, t)$ as the solution of the boundary-value problem

$$u_t = a^2(x)u_{xx} + g(x, t), \quad |x| < \infty, \quad x \neq 0, \quad t > 0,$$

$$u|_{x=0^-} - u|_{x=0^+} = \gamma, \quad k_- u_x|_{x=0^-} = k_{j+1} u_x|_{x=0^+}, \quad t \geq 0,$$

$$u|_{t=0} = f(x), \quad |x| < \infty, \tag{2.18}$$

where $a(x) = a_-$, $x < 0$, $a(x) = a_+$, $x > 0$, and $\gamma = \gamma_+ - \gamma_-$. The problem is equivalent to the following scalar RHP:

$$U^+(s) = -\frac{s^2 a_+^2 - p}{s^2 a_-^2 - p} U^-(s) - \frac{H^-(s) + H^+(s) + \beta_1 + s\beta_0}{s^2 a_-^2 - p}, \quad s \in L, \tag{2.19}$$

where

$$\beta_j = a_-^2 \frac{d^j}{dx^j} \hat{u}(0^-; p) - a_+^2 \frac{d^j}{dx^j} \hat{u}(0^+; p), \quad j = 0, 1,$$

$$U^-(s) = \int_0^\infty e^{-sx} \hat{u} dx, \quad U^+(s) = \int_{-\infty}^0 e^{-sx} \hat{u} dx,$$

$$H^-(s) = \int_0^\infty e^{-sx}[f(x) + \hat{g}(x; p)]dx, \quad H^+(s) = \int_{-\infty}^0 e^{-sx}[f(x) + \hat{g}(x; p)]dx. \tag{2.20}$$

The functions $U^\pm$ and $H^\pm$ are analytic in the half-planes $D^\pm$. The coefficient of the RHP is a rational function, and the functions $U^+(s)$ and $U^-(s)$ are recovered in the standard manner,

$$U^+(s) = \frac{1}{sa_- - \sqrt{p}}\left[-\frac{a_+ h_-}{\sqrt{p}(a_+ + a_-)} - A_0 - \frac{1}{sa_- + \sqrt{p}}\left(H^+(s) + \frac{a_- h_+(sa_+ - \sqrt{p})}{\sqrt{p}(a_+ + a_-)}\right)\right],$$

$$U^-(s) = \frac{1}{sa_+ + \sqrt{p}}\left[\frac{a_- h_+}{\sqrt{p}(a_+ + a_-)} - A_1 - \frac{1}{sa_+ - \sqrt{p}}\left(H^-(s) - \frac{a_+ h_-(sa_- + \sqrt{p})}{\sqrt{p}(a_+ + a_-)}\right)\right], \tag{2.21}$$

where

$$h_+ = H^+\left(-\frac{\sqrt{p}}{a_-}\right), \quad h_- = H^-\left(\frac{\sqrt{p}}{a_+}\right),$$

$$A_0 = \frac{\beta_0\sqrt{p} + a_+\beta_1}{\sqrt{p}(a_+ + a_-)}, \quad A_1 = \frac{\beta_0\sqrt{p} - a_-\beta_1}{\sqrt{p}(a_+ + a_-)}, \tag{2.22}$$

Notice that the points $s = \pm\sqrt{p}/a_\pm \in D^\mp$ are removable singularities of the functions $U^\mp(s)$. The derivation of representations for the functions $\beta_0(p)$ and $\beta_1(p)$ requires inversion of the Laplace transforms in (2.21). This implies

$$\hat{u}(x;p) = \left[\frac{2a_-h_+ - (a_+ - a_-)h_-}{2a_+(a_+ + a_-)\sqrt{p}} - \frac{A_1}{a_+}\right]e^{-\sqrt{p}x/a_+}$$

$$+\frac{1}{2a_+\sqrt{p}}\int_0^\infty [f(\xi) + \hat{g}(\xi;p)]e^{-\sqrt{p}|x-\xi|/a_+}d\xi, \quad 0 < x < \infty,$$

$$\hat{u}(x;p) = \left[\frac{2a_+h_- + (a_+ - a_-)h_+}{2a_-(a_+ + a_-)\sqrt{p}} + \frac{A_0}{a_-}\right]e^{\sqrt{p}x/a_-}$$

$$+\frac{1}{2a_-\sqrt{p}}\int_{-\infty}^0 [f(\xi) + \hat{g}(\xi;p)]e^{-\sqrt{p}|x-\xi|/a_-}d\xi, \quad -\infty < x < 0. \tag{2.23}$$

The function $\hat{u}(x;p)$ and its derivative $\hat{u}_x(x;p)$ are discontinuous at the point $x = 0$ and due to (2.18) have to meet the conditions

$$\hat{u}|_{x=0^-} - \hat{u}|_{x=0^+} = \frac{\gamma}{p}, \quad k_-\hat{u}_x|_{x=0^-} - k_+\hat{u}_x|_{x=0^+} = 0, \tag{2.24}$$

On satisfying these conditions we eventually determine the functions β_0 and β_1 as

$$\beta_0 = \frac{1}{a_+k_- + a_-k_+}\left[\frac{(a_-^2 - a_+^2)(a_+^2k_-h_+ + a_-^2k_+h_-)}{a_+a_-\sqrt{p}} + \frac{\gamma(a_+^3k_- + a_-^3k_+)}{p}\right],$$

$$\beta_1 = \frac{a_-^2k_+ - a_+^2k_-}{a_+k_- + a_-k_+}\left[\frac{a_-h_- - a_+h_+}{a_+a_-} + \frac{\gamma}{\sqrt{p}}\right]. \tag{2.25}$$

If we substitute these expressions in formulas (2.23), the expressions for the function $\hat{u}(x;p)$ are simplified and become

$$\hat{u}(x;p) = C_\pm(p)e^{\mp\sqrt{p}x/a_\pm} \pm \frac{1}{2a_\pm\sqrt{p}}\int_0^{\pm\infty}[f(\xi) + \hat{g}(\xi;p)]e^{-\sqrt{p}|x-\xi|/a_\pm}d\xi, \quad \pm x > 0. \tag{2.26}$$

Here,

$$C_+(p) = \frac{\lambda_1 h_-}{2a_+\sqrt{p}} + \frac{\lambda_- h_+}{a_-\sqrt{p}} + \frac{k_-\gamma}{\lambda_0(p-a)}, \quad C_-(p) = -\frac{\lambda_1 h_+}{2a_-\sqrt{p}} + \frac{\lambda_+ h_-}{a_+\sqrt{p}} + \frac{k_+\gamma}{\lambda_0(p+a)},$$

$$\lambda_1 = \frac{1}{\lambda_0}\left(\frac{k_+}{a_+} - \frac{k_-}{a_-}\right), \quad \lambda_\pm = \frac{k_\pm}{a_\pm\lambda_0}, \quad \lambda_0 = \frac{k_+}{a_+} + \frac{k_-}{a_-}. \tag{2.27}$$

To finalize our derivations, we apply the inverse Laplace transform and take into consideration the formulas

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-\sqrt{p}\alpha + pt}dp}{\sqrt{p}} = \frac{1}{\sqrt{\pi t}}e^{-\alpha^2/(4t)},$$

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$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-\sqrt{p}\alpha+pt}dp}{p}=\operatorname{Erfc}\left(\frac{\alpha}{2\sqrt{t}}\right),\quad \operatorname{Re}\alpha>0,\quad \operatorname{Re}c>0. \qquad (2.28)$$

Here, $\operatorname{Erfc}(\cdot)$ is the the complementary error function. This implies the following representations for the function $u(x,t)$ when x is negative:

$$u(x,t)=\frac{k_+\gamma}{\lambda_0 a_+}\operatorname{Erfc}\left(-\frac{x}{2\sqrt{t}a_-}\right)+\frac{\lambda_+}{a_+\sqrt{\pi t}}\int_0^\infty e^{-(x/a_--\xi/a_+)^2/(4t)}f(\xi)d\xi$$

$$+\frac{1}{2a_-\sqrt{\pi t}}\int_{-\infty}^0\left[-\lambda_1 e^{-(x+\xi)^2/(4a_-^2 t)}+e^{-(x-\xi)^2/(4a_-^2 t)}\right]f(\xi)d\xi$$

$$+\frac{\lambda_+}{a_+\sqrt{\pi}}\int_0^t\int_0^\infty e^{-(x/a_--\xi/a_+)^2/[4(t-\tau)]}\frac{g(\xi,\tau)d\xi d\tau}{\sqrt{t-\tau}}$$

$$+\frac{1}{2a_-\sqrt{\pi}}\int_0^t\int_{-\infty}^0\left[-\lambda_1 e^{-(x+\xi)^2/[4a_-^2(t-\tau)]}+e^{-(x-\xi)^2/[4a_-^2(t-\tau)]}\right]\frac{g(\xi,\tau)d\xi d\tau}{\sqrt{t-\tau}}. \qquad (2.29)$$

For x positive we have

$$u(x,t)=-\frac{k_-\gamma}{\lambda_0 a_-}\operatorname{Erfc}\left(\frac{x}{2\sqrt{t}a_+}\right)+\frac{\lambda_-}{a_-\sqrt{\pi t}}\int_{-\infty}^0 e^{-(x/a_+-\xi/a_-)^2/(4t)}f(\xi)d\xi$$

$$+\frac{1}{2a_+\sqrt{\pi t}}\int_0^\infty\left[\lambda_1 e^{-(x+\xi)^2/(4a_+^2 t)}+e^{-(x-\xi)^2/(4a_+^2 t)}\right]f(\xi)d\xi$$

$$+\frac{\lambda_-}{a_-\sqrt{\pi}}\int_0^t\int_{-\infty}^0 e^{-(x/a_+-\xi/a_-)^2/[4(t-\tau)]}\frac{g(\xi,\tau)d\xi d\tau}{\sqrt{t-\tau}}$$

$$+\frac{1}{2a_+\sqrt{\pi}}\int_0^t\int_0^\infty\left[\lambda_1 e^{-(x+\xi)^2/[4a_+^2(t-\tau)]}+e^{-(x-\xi)^2/[4a_+^2(t-\tau)]}\right]\frac{g(\xi,\tau)d\xi d\tau}{\sqrt{t-\tau}}. \qquad (2.30)$$

The total temperature $u_0(x,t)$ given by formula (2.17) is bounded and has different limits as $x\to\pm\infty$. When x is kept finite and $t\to\infty$, the temperature has a finite limit independent of x,

$$\lim_{t\to\infty}u_0(x,t)=\frac{\gamma_-a_+k_-+\gamma_+a_-k_+}{\lambda_0 a_+a_-},\quad -X_1<x<X_2. \qquad (2.31)$$

Here, X_1 and X_2 are any finite positive numbers. Formula (2.31) is consistent with the result obtained by Deconinck et al (2014). If $\gamma_-=\gamma_+$ ($\gamma=0$) and there is no heat source ($g(x,t)\equiv 0$), then the representations (2.29) and (2.30) generalize to the discontinuous case the classical Poisson formula obtained for an infinite homogeneous rod.

Notice that it is possible to bypass the RHP and derive the representation (2.26) for the function $\hat{u}(x;p)$ directly by employing the fundamental functions

$$\frac{1}{2a_\pm\sqrt{p}}e^{-\sqrt{p}|x-\xi|/a_\pm} \qquad (2.32)$$

of the differential operators $a_\pm^2\frac{d^2}{dx^2}-p$. The functions $C_\pm(p)$ are determined in the same manner as before from the two conditions (2.24). Their expressions coincide with those given by (2.27). It is evident that the same approach works for any number of discontinuities including the case of a finite discontinuous rod with any physical boundary conditions imposed at the ends. In this case the fundamental functions (2.32) need to be replaced by the corresponding Green functions of the one-dimensional boundary value problems. These Green functions are derived in an elementary fashion. If the number of discontinuities is $n\geq 3$, then instead of two functions $C_+(p)$ and $C_-(p)$ we have $n-1$ pairs of unknown functions. They are determined by a system of $2n-2$ linear algebraic equations following from the $2n-2$ conditions at the discontinuity points.

2.3 Abrahams-Wickham system of integral equations

Abrahams and Wickham (1990) analyzed the system

$$u(x) = \lambda \int_0^\infty k(x-t)u(t)dt + f(x), \quad 0 < x < \infty, \tag{2.33}$$

where the matrix-kernel is given by

$$k(x) = \begin{pmatrix} e^{-|x|} & e^{-|x-a|} \\ e^{-|x+a|} & e^{-|x|} \end{pmatrix}, \tag{2.34}$$

λ and a are parameters, $u(x) = (u_1(x), u_2(x))^\top$, and $f(x)$ is a forcing vector-function prescribed accordingly. To factorize the matrix coefficient of the RHP associated with the system (2.33), they expressed the matrix-factors through the solution of a certain auxiliary system of integral equations. In the case (2.34) that system admits an exact solution. In what follows we derive a closed-form solution by a simple method that bypasses not only the auxiliary system of integral equations, but also the matrix Wiener-Hopf factorization. First we apply the Fourier integral transform to the system (2.33) and have the following RHP on the real axis

$$G(\alpha)U^+(\alpha) = U^-(\alpha) + F^+(\alpha), \quad -\infty < \alpha < +\infty, \tag{2.35}$$

where

$$G(\alpha) = \frac{1}{\alpha^2 + 1} \begin{pmatrix} \alpha^2 + 1 - 2\lambda & -2\lambda e^{i\alpha a} \\ -2\lambda e^{-i\alpha a} & \alpha^2 + 1 - 2\lambda \end{pmatrix}, \quad F^+(\alpha) = \int_0^\infty f(x)e^{i\alpha x}dx,$$

$$U^+(\alpha) = \int_0^\infty u(x)e^{i\alpha x}dx, \quad U^-(\alpha) = \int_{-\infty}^0 u_-(x)e^{i\alpha x}dx, \tag{2.36}$$

the vector-functions $U^\pm(\alpha) = (U_1^\pm(\alpha), U_2^\pm(\alpha))^\top$ are analytic in the half-planes $C^\pm = \pm \operatorname{Im} \alpha > 0$, the vector-function $F^+(\alpha) = (F_1^+(\alpha), F_2^+(\alpha))^\top$ is analytic in the upper half-plane C^+, and the vector-function $u_-(x)$, $x < 0$, is given by $u_-(x) = \lambda \int_0^\infty k(x-t)u(t)dt$.

Next, instead of factorizing the matrix $G(\alpha)$ we express the function $U_1^+(\alpha)$ from the first equation in (2.35) and substitute it into the second equation. After obvious simplifications this brings us to the new system of functional equations

$$\frac{\alpha^2 + 1 - 2\lambda}{\alpha^2 + 1}U_1^+(\alpha) - \frac{2\lambda e^{i\alpha a}}{\alpha^2 + 1}U_2^+(\alpha) = F_1^+(\alpha) + U_1^-(\alpha),$$

$$\frac{\alpha^2 + 1 - 2\lambda}{\alpha^2 + 1 - 4\lambda}[U_2^-(\alpha) + F_2^+(\alpha)] + \frac{2\lambda e^{-i\alpha a}}{\alpha^2 + 1 - 4\lambda}[U_1^-(\alpha) + F_1^+(\alpha)] = U_2^+(\alpha). \tag{2.37}$$

Notice that the products $e^{i\alpha a}U_2^+(\alpha)$ and $e^{-i\alpha a}U_1^-(\alpha)$ are analytic in the domains C^+ and C^-, respectively, and vanish exponentially when $\alpha \to \infty$ and $\alpha \in C^\pm$. For simplicity, we analyze further the normal case that is we assume that $\lambda \notin [1/4, +\infty)$. Choose $\arg(1-4\lambda) \in (-\pi, \pi)$, $\arg(1-2\lambda) \in (-\pi, \pi)$, and denote $\lambda_0 = \sqrt{1-2\lambda}$, $\lambda_1 = \sqrt{1-4\lambda}$, $\arg \lambda_j \in (-\pi/2, \pi/2)$, $j = 0, 1$. Factorize now the rational functions

$$\frac{\alpha^2 + 1 - 2\lambda}{\alpha^2 + 1} = \frac{K_0^+(\alpha)}{K_0^-(\alpha)}, \quad \frac{\alpha^2 + 1 - 2\lambda}{\alpha^2 + 1 - 4\lambda} = \frac{K_1^+(\alpha)}{K_1^-(\alpha)}, \tag{2.38}$$

where
$$K_0^{\pm}(\alpha) = \left(\frac{\alpha \pm i\lambda_0}{\alpha \pm i}\right)^{\pm 1}, \quad K_1^{\pm}(\alpha) = \left(\frac{\alpha \pm i\lambda_0}{\alpha \pm i\lambda_1}\right)^{\pm 1}. \tag{2.39}$$

In the general case of the forcing vector-function $f(x)$ we need to introduce new functions, $\Psi_1(\alpha)$ and $\Psi_2(\alpha)$, the Cauchy integrals with the density chosen according to the following relations for their limit values on the real axis

$$\Psi_1^+(\alpha) - \Psi_1^-(\alpha) = F_1^+(\alpha)K_0^-(\alpha), \quad -\infty < \alpha < +\infty,$$

$$\Psi_2^+(\alpha) - \Psi_2^-(\alpha) = \frac{F_2^+(\alpha)}{K_1^-(\alpha)} + \frac{2\lambda e^{-i\alpha a}F_1^+(\alpha)}{(\alpha + i\lambda_0)(\alpha - i\lambda_1)}, \quad -\infty < \alpha < +\infty. \tag{2.40}$$

We seek the solution $U_j^{\pm}(\alpha)$ in the class of functions vanishing at infinity. Therefore the subsequent application of the generalized Liouville theorem enables us to determine the solution of the RHP in the form

$$U_1^-(\alpha) = \frac{1}{K_0^-(\alpha)}\left[\Psi_1^-(\alpha) + \frac{C_1}{\alpha - i\lambda_0}\right],$$

$$U_2^+(\alpha) = K_1^+(\alpha)\left[\Psi_2^+(\alpha) + \frac{C_2}{\alpha + i\lambda_0}\right],$$

$$U_2^-(\alpha) = K_1^-(\alpha)\left[\Psi_2^-(\alpha) + \frac{C_2}{\alpha + i\lambda_0} - \frac{2\lambda e^{-ia\alpha}}{(\alpha + i\lambda_0)(\alpha - i\lambda_1)K_0^-(\alpha)}\left(\Psi_1^-(\alpha) + \frac{C_1}{\alpha - i\lambda_0}\right)\right],$$

$$U_1^+(\alpha) = \frac{1}{K_0^+(\alpha)}\left[\Psi_1^+(\alpha) + \frac{C_1}{\alpha - i\lambda_0} + \frac{2\lambda e^{ia\alpha}K_1^+(\alpha)}{(\alpha + i)(\alpha - i\lambda_0)}\left(\Psi_2^+(\alpha) + \frac{C_2}{\alpha + i\lambda_0}\right)\right]. \tag{2.41}$$

Here, C_1 and C_2 are arbitrary constants. It is easy to observe that the functions $U_1^+(\alpha)$ and $U_2^-(\alpha)$ have inadmissible simple poles at the points $\alpha = i\lambda_0 \in C^+$ and $\alpha = -i\lambda_0 \in C^-$, respectively. They can be removed if the residues of the functions $U_1^+(\alpha)$ and $U_2^-(\alpha)$ at these points vanish. Simple calculations give the following expressions for the constants C_1 and C_2:

$$C_1 = \frac{d_1 + bd_2}{b^2 + 1}, \quad C_2 = \frac{d_2 - bd_1}{b^2 + 1}, \tag{2.42}$$

where
$$b = \frac{2\lambda e^{-a\lambda_0}}{(\lambda_0 + 1)(\lambda_0 + \lambda_1)},$$

$$d_1 = 2ib\lambda_0\Psi_2^+(i\lambda_0), \quad d_2 = 2ib\lambda_0\Psi_1^-(-i\lambda_0). \tag{2.43}$$

The inverse Fourier transformation of formulas (2.41) for $U_1^+(\alpha)$ and $U_2^+(\alpha)$ yields the solution of the original system of integral equations (2.33).

3 Laplace equation in a wedge with mixed boundary conditions: the case of meromorphic functions with periodic poles and zeros

As an illustration of the method in the case when the entries of the matrix coefficient of the RHP are almost periodic functions and meromorphic functions with periodic poles

and zeros, we consider the following mixed boundary value problem for the Laplace operator:

$$\Delta u(r,\theta) = 0, \quad 0 < r < \infty, \quad 0 < \theta < \alpha,$$

$$u|_{\theta=0} = f_{1-}(r), \quad 0 < r < a_1; \quad -\frac{\partial u}{r\partial\theta}\bigg|_{\theta=0} = f_{1+}(r), \quad r > a_1,$$

$$u|_{\theta=\alpha} = f_{2-}(r), \quad 0 < r < a_2; \quad \frac{\partial u}{r\partial\theta}\bigg|_{\theta=\alpha} = f_{2+}(r), \quad r > a_2. \tag{3.1}$$

The function u is sought in the class of functions bounded at $r = 0$ and vanishing at $r = \infty$ as $|u(r,\theta)| \le A r^{-\beta}$, $0 \le \theta \le \alpha$, $A = \text{const}$, $0 < \beta < \varepsilon$, and ε is a small positive number. This problem can be interpreted as a model of stationary heat conduction if $u^{(t)}(r,\theta) = u(r,\theta) + T_\infty$ is the temperature and T_∞ is the temperature at infinity. T_∞ is constant for all $\theta \in [0,\alpha]$ and has to be determined a posteriori. The temperature $u^{(t)} = T_1$ is prescribed in the segment $0 < r < a_1$, $\theta = 0$ as $T_1(r) = f_{1-}(r) + T_\infty$ and in the segment $0 < r < a_2$, $\theta = \alpha$ as $u^{(t)} = T_2(r) = f_{2-}(r) + T_\infty$. In the rest of the boundary, the heat flux q is given: $q = k_- f_{1+}(r)$, $r > a_1$, $\theta = 0$, and $q = k_+ f_{2+}(r)$, $r > a_2$, $\theta = \alpha$. (k_- and k_+ are the thermal conductivities of the lower and upper boundaries, respectively). It is also assumed that the functions $f_{j+}(r)$ decay at infinity as $f_{j+}(r) = O(r^{-1-\beta})$, $r \to \infty$, $j = 1, 2$.

3.1 Vector RHP

Before proceeding with the solution, we extend all the boundary conditions for the whole semi-axis as

$$u|_{\theta=0} = f_{1-}(r) + \varphi_{1+}(r), \quad -\frac{\partial u}{r\partial\theta}\bigg|_{\theta=0} = \varphi_{1-}(r) + f_{1+}(r), \quad 0 < r < \infty,$$

$$u|_{\theta=\alpha} = f_{2-}(r) + \varphi_{2+}(r), \quad \frac{\partial u}{r\partial\theta}\bigg|_{\theta=\alpha} = \varphi_{2-}(r) + f_{2+}(r), \quad 0 < r < \infty, \tag{3.2}$$

where

$$\text{supp}\, f_{j-}(r) \subset [0, a_j], \quad \text{supp}\, f_{j+}(r) \subset [a_j, \infty),$$

$$\text{supp}\, \varphi_{j-}(r) \subset [0, a_j], \quad \text{supp}\, \varphi_{j+}(r) \subset [a_j, \infty), \tag{3.3}$$

and the functions $\varphi_{j\pm}(r)$ need to be recovered. We shall now apply the Mellin transformation to the Laplace equation and the extended boundary conditions (3.2). Let

$$F_j^-(s) = \int_0^1 f_j(a_j\rho)\rho^{s-1}d\rho, \quad F_j^+(s) = a_j\int_1^\infty f_j(a_j\rho)\rho^s d\rho,$$

$$\Phi_j^-(s) = a_j\int_0^1 \varphi_{j-}(a_j\rho)\rho^s d\rho, \quad \Phi_j^+(s) = \int_1^\infty \varphi_{j+}(a_j\rho)\rho^{s-1}d\rho, \tag{3.4}$$

and

$$\hat{u}_s(\theta) = \int_0^\infty u(r,\theta)r^{s-1}dr, \quad s \in L : \text{Re}\, s = \sigma \in (0,\beta). \tag{3.5}$$

The function $\hat{u}_s(\theta)$ solves the differential equation $\hat{u}_s''(\theta) + s^2\hat{u}_s(\theta) = 0$ and has the form

$$\hat{u}_s(\theta) = A_1(s)\cos s\theta + A_2(s)\sin s\theta, \tag{3.6}$$

where A_1 and A_2 are fixed from the first two boundary conditions in (3.2) as

$$A_1 = a_1^s[\Phi_1^+(s) + F_1^-(s)], \quad A_2 = -\frac{a_1^s}{s}[\Phi_1^-(s) + F_1^+(s)]. \tag{3.7}$$

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The third and fourth boundary conditions in (3.2) constitute the vector RHP

$$\Phi^+(s) = -\frac{1}{s}G(s)[\Phi^-(s) + F^+(s)] - F^-(s), \quad s \in L, \tag{3.8}$$

where

$$G(s) = \begin{pmatrix} \cot \alpha s & \lambda^s \csc \alpha s \\ \lambda^{-s} \csc \alpha s & \cot \alpha s \end{pmatrix}, \quad \lambda = \frac{a_2}{a_1}. \tag{3.9}$$

Without loss of generality we assume that $\lambda > 1$. The vectors $\Phi^\pm(s) = (\Phi_1^\pm(s), \Phi_2^\pm(s))^\top$ are analytic in the half-planes $D^\pm : \pm\operatorname{Re} s < \pm\sigma$, $\sigma \in (0, \beta)$, and $0 \in D^+$.

3.2   Factorization of the matrix $G(s)$

We aim to find two matrices, $X^+(s)$ and $X^-(s)$, analytic in the domains D^+ and $D^-(s)$, respectively, having a finite order at infinity and solving the following matrix equation

$$X^+(s) = G(s)X^-(s), \quad s \in L. \tag{3.10}$$

Denote

$$X^\pm(s) = \begin{pmatrix} \chi_{11}^\pm(s) & \chi_{12}^\pm(s) \\ \chi_{21}^\pm(s) & \chi_{22}^\pm(s) \end{pmatrix}. \tag{3.11}$$

Regrouping terms in the same fashion as in section 2.1 we obtain

$$\chi_{1j}^+(s) = -\tan \alpha s \chi_{1j}^-(s) + \frac{\lambda^s \chi_{2j}^+(s)}{\cos \alpha s},$$

$$\chi_{2j}^-(s) = \tan \alpha s \chi_{2j}^+(s) - \frac{\lambda^{-s} \chi_{1j}^-(s)}{\cos \alpha s}. \tag{3.12}$$

It is seen that the functions $\chi_{1j}^+(s)$ and $\chi_{2j}^-(s)$ have simple poles at the points $s = -\frac{\pi}{\alpha}(n + \frac{1}{2}) \in D^+$ and $s = \frac{\pi}{\alpha}(n + \frac{1}{2}) \in D^-$ $(n = 0, 1, \ldots)$, respectively, and they have to be removed. We seek the functions $\chi_{1j}^-(s)$ and $\chi_{2j}^+(s)$ in the form of the hypergeometric series

$$\chi_{1j}^-(s) = \sum_{k=0}^\infty \frac{\Gamma(\alpha s/\pi + k + \nu_1)\Gamma(k + \nu_2)}{\Gamma(\alpha s/\pi + k + \mu_1 + 1/2)\Gamma(k + \mu_2)}(-1)^k \lambda^{-k\pi/\alpha + \sigma_1},$$

$$\chi_{2j}^+(s) = \kappa \sum_{k=0}^\infty \frac{\Gamma(-\alpha s/\pi + k + \alpha_1)\Gamma(k + \alpha_2)}{\Gamma(-\alpha s/\pi + k + \beta_1 + 1/2)\Gamma(k + \beta_2)}(-1)^k \lambda^{-k\pi/\alpha + \sigma_2}. \tag{3.13}$$

The parameters ν_j, μ_j, α_j, β_j, σ_j $(j = 1, 2)$, and κ are to be determined from the following conditions:

(i) the functions $\chi_{1j}^+(s)$ have removable singularities at the zeros of $\cos \alpha s$ lying in the half-plane D^+,

$$-\sin \alpha s \chi_{1j}^-(s) + \lambda^s \chi_{2j}^+(s) = 0, \quad s = -\frac{\pi}{\alpha}\left(n + \frac{1}{2}\right), \quad n = 0, 1, \ldots, \tag{3.14}$$

(ii) the functions $\chi_{2j}^-(s)$ have removable singularities at the points $s = \frac{\pi}{\alpha}(n + \frac{1}{2}) \in D^-$

$$\sin \alpha s \chi_{2j}^+(s) - \lambda^{-s} \chi_{1j}^-(s) = 0, \quad s = \frac{\pi}{\alpha}\left(n + \frac{1}{2}\right), \quad n = 0, 1, \ldots, \tag{3.15}$$

(iii) the functions $\chi_{1j}^-(s)$ may have simple poles at the zeros of $\tan \alpha s$ lying in D^+, $s = -\pi n/\alpha$, $n = 0,1,\ldots$; it follows from (3.12) that then the functions $\chi_{1j}^+(s)$ have removable singularities at these points,

(iv) the functions $\chi_{2j}^+(s)$ may have simple poles at the points $s = \pi n/\alpha \in D^-$, $n = 1,2,\ldots$. Then the functions $\chi_{2j}^-(s)$ have removable singularities at these points.

The conditions (i) and (ii) when satisfied give

$$\beta_2 = 1, \quad \mu_2 = 1, \quad \beta_1 = 1 - \mu_1, \quad \alpha_2 = 1/2 + \nu_1 - \mu_1,$$

$$\nu_2 = \alpha_1 + \mu_1 - 1/2, \quad \kappa = (-1)^{\mu_1}, \quad \sigma_2 = \sigma_1 + \frac{\pi(2\mu_1 - 1)}{2\alpha}. \tag{3.16}$$

The other two conditions, (iii) and (iv), determine ν_1 and α_1: $\nu_1 = 0$ and $\alpha_1 = 1$. Without loss of generality, $\sigma_1 = 0$. This brings us to the following one-parametric family of solutions:

$$\chi_{1j}^-(s) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha s/\pi + k)\Gamma(k + \mu_1 + 1/2)(-1)^k \lambda^{-k\pi/\alpha}}{\Gamma(\alpha s/\pi + k + \mu_1 + 1/2)k!},$$

$$\chi_{2j}^+(s) = \sum_{k=0}^{\infty} \frac{\Gamma(-\alpha s/\pi + k + 1)\Gamma(k - \mu_1 + 1/2)(-1)^{k-\mu_1} \lambda^{(\mu_1 - k - 1/2)\pi/\alpha}}{\Gamma(-\alpha s/\pi + k - \mu_1 + 3/2)k!}. \tag{3.17}$$

On choosing $\mu_1 = 0$ for $j = 1$ we obtain the functions $\chi_{11}^-(s)$ and $\chi_{21}^+(s)$ vanishing at infinity as $\chi_{11}^-(s) = O(s^{-1/2})$, $s \in D^-$, and $\chi_{21}^+(s) = O(s^{-1/2})$, $s \in D^+$, and admitting the following representations through the beta and Gauss functions:

$$\chi_{11}^-(s) = B\left(\frac{\alpha s}{\pi}, \frac{1}{2}\right) F\left(\frac{\alpha s}{\pi}, \frac{1}{2}; \frac{\alpha s}{\pi} + \frac{1}{2}; -\lambda^{-\pi/\alpha}\right),$$

$$\chi_{21}^+(s) = \lambda^{-\pi/(2\alpha)} B\left(-\frac{\alpha s}{\pi} + 1, \frac{1}{2}\right) F\left(-\frac{\alpha s}{\pi} + 1, \frac{1}{2}; -\frac{\alpha s}{\pi} + \frac{3}{2}; -\lambda^{-\pi/\alpha}\right). \tag{3.18}$$

The function $\chi_{11}^-(s)$ is analytic in the half-plane D^- and meromorphic in D^+, while the function $\chi_{21}^+(s)$ is analytic in D^+ and meromorphic in D^-. In the case $j = 2$, we put $\mu_1 = 1$ and derive the other two functions

$$\chi_{12}^-(s) = B\left(\frac{\alpha s}{\pi}, \frac{3}{2}\right) F\left(\frac{\alpha s}{\pi}, \frac{3}{2}; \frac{\alpha s}{\pi} + \frac{3}{2}; -\lambda^{-\pi/\alpha}\right),$$

$$\chi_{22}^+(s) = -\lambda^{\pi/(2\alpha)} B\left(-\frac{\alpha s}{\pi} + 1, -\frac{1}{2}\right) F\left(-\frac{\alpha s}{\pi} + 1, -\frac{1}{2}; -\frac{\alpha s}{\pi} + \frac{1}{2}; -\lambda^{-\pi/\alpha}\right). \tag{3.19}$$

Notice that the function $\chi_{12}^-(s)$ vanishes at infinity, while the function $\chi_{22}^+(s)$ grows as $s \to \infty$: $\chi_{12}^-(s) = O(s^{-3/2})$, $\chi_{22}^+(s) = O(s^{1/2})$.

The functions $\chi_{1j}^+(s)$ and $\chi_{2j}^-(s)$ are expressed through the functions (3.18) and (3.19) by formulas (3.12). They are analytic in D^+ and D^-, and their order at infinity is determined by the order of $\chi_{1j}^-(s)$ and $\chi_{2j}^+(s)$, respectively. This completes the exact factorization of the matrix $G(s)$. The factorization we found is not unique: by choosing the parameter μ_1 as an integer different from 0 and 1 we can construct another set of the functions $\chi_{12}^-(s)$ and $\chi_{22}^+(s)$ and that is why another factorization different from the one found. The Wiener-Hopf matrix factors $X^+(s)$ and X^- given by (3.18) and (3.19) have the following asymptotics at infinity:

$$X^\pm(s) = \begin{pmatrix} O(s^{-1/2}) & O(s^{-3/2}) \\ O(s^{-1/2}) & O(s^{1/2}) \end{pmatrix}, \quad s \to \infty. \tag{3.20}$$

3.3 Exact solution of the heat conduction problem in a wedge

On having factorized the matrix $G(s)$ we replace $G(s)$ by the product $X^+(s)[X^-(s)]^{-1}$ in the boundary condition (3.8) of the vector RHP

$$[X^+(s)]^{-1}[\Phi^+(s) + F^-(s)] = -\frac{1}{s}[X^-(s)]^{-1}[\Phi^-(s) + F^+(s)], \quad s \in L. \tag{3.21}$$

Represent next the functions $f_{1-}(r)$ and $f_{2-}(r)$ as

$$f_{j-}(r) = \tilde{T}_j + T_j^*(r), \quad \tilde{T}_j = -T_\infty + T_j^\circ, \quad j = 1, 2, \tag{3.22}$$

where $T_j^\circ + T_j^*(r) = T_j(r)$ is the prescribed temperature in the lower boundary $0 < r < a_1$ and the upper boundary $0 < r < a_2$ of the wedge, $T_j^\circ = \text{const}$, and $T_j^*(r) = O(r^{\gamma_j})$, $r \to 0$, $\gamma_j > 0$. We also introduce the Mellin transforms

$$\hat{T}_j^-(s) = \int_0^1 T_j^*(a_j\rho)\rho^{s-1}d\rho, \quad j = 1, 2, \tag{3.23}$$

and the Cauchy integrals

$$\Psi(s) = \frac{1}{2\pi i}\int_L \frac{[X^+(\sigma)]^{-1}\hat{T}^-(\sigma)d\sigma}{\sigma - s},$$

$$\Omega(s) = \frac{1}{2\pi i}\int_L \frac{[X^-(\sigma)]^{-1}F^+(\sigma)d\sigma}{\sigma(\sigma - s)}. \tag{3.24}$$

Employ now the continuity principle and the Liouville theorem and derive the following representations of the solution:

$$\Phi^+(s) = \left(-I + X^+(s)[X^+(0)]^{-1}\right)\frac{\tilde{T}}{s} - X^+(s)[\Psi^+(s) + \Omega^+(s)],$$

$$\Phi^-(s) = -X^-(s)[X^+(0)]^{-1}\tilde{T} + sX^-(s)[\Psi^-(s) + \Omega^-(s)], \tag{3.25}$$

where $\tilde{T} = (\tilde{T}_1, \tilde{T}_2)^\top$. Further, since the functions $\chi_{22}^\pm(s)$ are growing at infinity, and the matrix factors have the asymptotics (3.20), the second components of the vectors $\Phi^-(s)$ and $\Phi^+(s)$, in general, have the asymptotics $\Phi_2^-(s) = O(s^{1/2})$, $\Phi_2^+(s) = O(s^{-1/2})$. Due to the Tauberian theorems for the Mellin transforms this causes an infinite temperature as $r \to a_j^+$ at the lower ($j = 1$) and the upper ($j = 2$) boundaries. Also, as $r \to a_j^-$, the heat flux has a nonintegrable singularity of order $3/2$. To derive the condition which guaranties the boundedness of the temperature and integrability of the heat flux, we denote the limits

$$\Psi^\circ = (\Psi_1^\circ, \Psi_2^\circ)^\top = \lim_{s \to \infty, s \in D^\pm} s\Psi^\pm(s),$$

$$\Omega^\circ = (\Omega_1^\circ, \Omega_2^\circ)^\top = \lim_{s \to \infty, s \in D^\pm} s\Omega^\pm(s), \tag{3.26}$$

and compute $[X^+(0)]^{-1}\tilde{T}$. It is directly verified that

$$\chi_{11}^+(0) = -\pi + \chi_{21}^+(0), \quad \chi_{12}^+(0) = -\pi + \chi_{22}^+(0), \tag{3.27}$$

and

$$\chi_{21}^+(0) = 2\tan^{-1}\lambda^{-\pi/(2\alpha)}, \quad \chi_{22}^+(0) = 2\lambda^{\pi/(2\alpha)} + 2\tan^{-1}\lambda^{-\pi/(2\alpha)}. \tag{3.28}$$

Continuing, the vectors $\Phi^{\pm}(s)$ have the asymptotics at infinity we need if and only if

$$-\tilde{T}_1 + \tilde{T}_2\left(1 - \frac{\pi}{\chi_{21}^+(0)}\right) = \frac{\Delta^+(0)}{\chi_{21}^+(0)}(\Psi_2^\circ + \Omega_2^\circ), \tag{3.29}$$

where $\Delta^+(0) = \det X^+(0) = -2\pi\lambda^{\pi/(2\alpha)}$. If this condition is fulfilled, then as $s \to \infty$, $\Phi^-(s) = O(s^{-1/2})$, $s \in D^-$, and $\Phi^+(s) = O(s^{-1})$, $s \in D^+$. The condition can be easily satisfied by the corresponding choice of the parameter T_∞ undetermined at this stage. Since $\tilde{T}_j = -T_\infty + T_j^\circ$, we transform equation (3.29) and find the temperature at infinity

$$T_\infty = \frac{2}{\pi}\tan^{-1}\lambda^{-\pi/(2\alpha)}(T_1^\circ - T_2^\circ) + T_2^\circ - 2\lambda^{\pi/(2\alpha)}(\Psi_2^\circ + \Omega_2^\circ). \tag{3.30}$$

In particular, if $T_j^*(r) = 0$ and $f_{j+}(r) = 0$, then $\Psi_2^\circ = 0$ and $\Omega_2^\circ = 0$. On assuming further that $T_1^\circ = 0$ (the case considered by Abrahams and Wickham (1990)), we obtain the simple explicit formula

$$T_\infty = \left[-\frac{2}{\pi}\tan^{-1}\lambda^{-\pi/(2\alpha)} + 1\right]T_2^\circ. \tag{3.31}$$

Finally, if $a_1 = a_2$, then $\lambda = 1$ and therefore $T_\infty = \frac{1}{2}T_2^\circ$. The same result for the case $\lambda = 1$ was derived by Abrahams and Wickham (1990).

We wish also to recover the function $u(r,\theta)$ and study its behavior as $r \to \infty$ and $r \to 0$. On applying the inverse Mellin transform to (3.6) and using (3.7) and (3.8) it is possible to have

$$u(r,\theta) = -\frac{1}{2\pi i}\int_L\left\{[\Phi_1^-(s) + F_1^+(s)]\cos(\alpha - \theta)s\left(\frac{r}{a_1}\right)^{-s}\right.$$

$$\left. + [\Phi_2^-(s) + F_2^+(s)]\cos\theta s\left(\frac{r}{a_2}\right)^{-s}\right\}\frac{ds}{s\sin\alpha s}. \tag{3.32}$$

Denote by κ_j those singular points of the functions $F_j^+(s)$ in the domain D^- which have the smallest real part among the singular points of $F_j^+(s)$ in D^-. Then we can derive

$$u(r,\theta) = O(r^{-\beta}), \quad r \to \infty, \quad \beta = \min\{\pi/\alpha, \kappa_1, \kappa_2\} > 0. \tag{3.33}$$

In particular, when $f_{1+}(r) = f_{2+}(r) = 0$ and $r \to \infty$,

$$u(r,\theta) = \frac{1}{\pi}\cos\frac{\pi\theta}{\alpha}\left[\Phi_1^-\left(\frac{\pi}{\alpha}\right)\left(\frac{r}{a_1}\right)^{-\pi/\alpha} - \Phi_2^-\left(\frac{\pi}{\alpha}\right)\left(\frac{r}{a_2}\right)^{-\pi/\alpha}\right] + O(r^{-2\pi/\alpha}), \tag{3.34}$$

and as $\theta = \alpha/2$, $u \sim cr^{-2\pi/\alpha}$, $r \to \infty$, c is a nonzero constant.

If we want to determine the behavior of the function u as $r \to 0$, we need regroup the terms in the integrand (3.32) in order to get rid of the functions $\Phi_1^-(s)$ and $\Phi_2^-(s)$. We have

$$u(r,\theta) = \frac{1}{2\pi i}\int_L\left\{[\Phi_1^+(s) + F_1^-(s)]\sin(\alpha - \theta)s\left(\frac{r}{a_1}\right)^{-s}\right.$$

$$\left. + [\Phi_2^+(s) + F_2^-(s)]\sin\theta s\left(\frac{r}{a_2}\right)^{-s}\right\}\frac{ds}{\sin\alpha s}. \tag{3.35}$$

Accepting the representations (3.22) by the Cauchy theorem we derive

$$u(r, \theta) \sim \left(1 - \frac{\theta}{\alpha}\right) T_1^\circ + \frac{\theta}{\alpha} T_2^\circ - T_\infty, \quad r \to 0. \tag{3.36}$$

Finally, we notice that the solution we obtained have the following asymptotics at the points where the type of the boundary conditions is changed:

$$u \sim c_j, \quad r \to a_j^+, \quad \frac{\partial u}{\partial \theta} = O(r^{-1/2}), \quad r \to a_j^-, \quad \theta = \begin{cases} \theta = 0, & j = 1, \\ \theta = \alpha, & j = 2. \end{cases} \tag{3.37}$$

4 Helmholtz equation in a strip: nonperiodic poles and zeros

With the example of the Helmholtz equation in a strip with a cut inside we shall show how the method of the RHP can be generalized when the zeros and poles of the meromorphic entries of the RHP matrix coefficient are not periodic. Consider the Neumann boundary value problem for the Helmholtz equation in the doubly connected domain $\Pi \setminus S$ with Π being a strip, $\Pi = \{|x_1| < \infty, -b_-^\circ < x_2 < b_+^\circ\}$, and S being a cut, $S = \{0 < x_1 < a, x_2 = 0^\pm\}$,

$$(\Delta + k_0^2)u^\circ = 0, \quad (x_1, x_2) \in \Pi \setminus S,$$

$$\frac{\partial u^\circ}{\partial x_2} = 0, \quad |x_1| < \infty, \quad x_2 = -b_-^\circ, b_+^\circ,$$

$$\frac{\partial u^\circ}{\partial x_2} = f^\circ(x_1), \quad 0 < x_1 < a, \quad x_2 = 0^\pm. \tag{4.1}$$

This problem can be interpreted as an antiplane problem for a strip Π with a crack S when the strip boundary is free of traction and the crack faces are subjected to the oscillating load $\tau_{23} = e^{i\omega t} G f^\circ$, t is time, ω is the frequency, and G is the shear modulus. This problem also describes acoustic wave propagation in a guideline with an acoustically hard screen S inside.

We assume that $\text{Im} \, k_0 > 0$. It will be convenient to work in the dimensionless coordinates $x = x_1/a$, $y = x_2/a$. Denote $k = ak_0$, $b_\pm = b_\pm^\circ/a$, $f(x) = af^\circ(ax)$, $u(x, y) = u^\circ(ax, ay)$. This transforms the problem to

$$(\Delta + k^2)u = 0, \quad (x, y) \in \{|x| < \infty, -b_- < y < b_+\} \setminus \{0 < x < 1, y = 0^\pm\},$$

$$\frac{\partial u}{\partial y} = 0, \quad |x| < \infty, \quad y = \pm b_\pm,$$

$$\frac{\partial u}{\partial y} = f(x), \quad 0 < x < 1, \quad y = 0^\pm, \quad u(x, 0^+) - u(x, 0^-) = \varphi_1(x), \quad |x| < \infty, \tag{4.2}$$

where $\text{supp} \, \varphi_1 \subset [0, 1]$ and $\varphi_1(x)$ is an unknown function in the segment $[0, 1]$.

On applying the Fourier transform with respect to x

$$\hat{u}_\alpha(y) = \int_{-\infty}^{\infty} u(x, y)e^{i\alpha x} dx \tag{4.3}$$

to the Helmholtz equation we easily obtain

$$\hat{u}_\alpha(y) = \begin{cases} C_1^+ \cosh \gamma(y - b_+) + C_2^+ \sinh \gamma(y - b_+), & 0 < y < b_+, \\ C_1^- \cosh \gamma(y + b_-) + C_2^- \sinh \gamma(y + b_-), & -b_- < y < 0, \end{cases} \tag{4.4}$$

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where $\gamma = \sqrt{\alpha^2 - k^2}$ is the single branch of the algebraic function $\gamma^2 = \alpha^2 - k^2$ fixed by the condition $\mathrm{Re}\,\gamma \geq 0$ or, equivalently, $\gamma(0) = -i$ in the α-plane cut along the straight line joining the branch points k and $-k$ and passing through the infinite point. The boundary conditions on the sides $y = \pm b_{\pm}$ and the discontinuity of u and continuity of its normal derivative in the line $y = 0$ yield

$$C_2^+ = C_2^- = 0, \quad C_1^+ = \frac{\sinh \gamma b_- \Phi_1^+(\alpha)}{\sinh \gamma (b_+ + b_-)}, \quad C_1^- = -\frac{\sinh \gamma b_+ \Phi_1^+(\alpha)}{\sinh \gamma (b_+ + b_-)}, \tag{4.5}$$

where

$$\Phi_1^+(\alpha) = \int_0^1 \varphi_1(x) e^{i\alpha x} dx. \tag{4.6}$$

To derive the RHP, we first extend the Neumann boundary condition on the line $y = 0$

$$\frac{\partial u}{\partial y} = f_-(x) + \varphi_{2-}(x) + \varphi_{2+}(x), \quad -\infty < x < \infty, \quad y = 0^{\pm}, \tag{4.7}$$

where $f_-(x) = f(x)$ if $0 < x < 1$ and it vanishes otherwise, $\varphi_{\pm}(x)$ are unknown functions such that $\mathrm{supp}\,\phi_{2-} \subset (-\infty, 0]$ and $\mathrm{supp}\,\phi_{2+} \subset [1, \infty)$. In order to apply the Fourier transform to the boundary condition (4.7), we introduce the following integrals

$$F^+(\alpha) = \int_0^1 f(x) e^{i\alpha x} dx,$$

$$\Phi_2^-(\alpha) = \int_{-\infty}^0 \varphi_{2-}(x) e^{i\alpha x} dx, \quad \Phi_2^+(\alpha) = \int_0^{\infty} \varphi_{2+}(x+1) e^{i\alpha x} dx. \tag{4.8}$$

We assert that $\Phi_1^+(\alpha)$ and $F^+(\alpha)$ are entire functions which are analytic in the upper half-plane and have an essential singularity at the infinite point in the lower half-plane and also

$$\Phi_1^+(\alpha) = e^{i\alpha} \Phi_1^-(\alpha), \quad F^+(\alpha) = e^{i\alpha} F^-(\alpha), \tag{4.9}$$

where

$$\Phi_1^-(\alpha) = \int_{-1}^0 \varphi_1(x+1) e^{i\alpha x} dx, \quad F^-(\alpha) = \int_{-1}^0 f(x+1) e^{i\alpha x} dx. \tag{4.10}$$

This implies that the boundary condition (4.7) is equivalent to the following vector RHP:

$$\Phi^+(\alpha) = G(\alpha) \Phi^-(\alpha) + F^-(\alpha) J, \quad -\infty < \alpha < +\infty. \tag{4.11}$$

Here,

$$G(\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ -g(\alpha) & -e^{-i\alpha} \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

$$g(\alpha) = \gamma \sinh \gamma b_+ \sinh \gamma b_- \mathrm{csch}\, \gamma (b_+ + b_-). \tag{4.12}$$

The new feature here is that the zeros and poles of the meromorphic entry of the matrix $G(\alpha)$, the function $g(\alpha)$, are not periodic, and we will pursue the RHP in a way different from the one proposed in the previous section. First we factorize the meromorphic function $g(\alpha)$

$$g(\alpha) = \frac{K^+(\alpha)}{K^-(\alpha)}, \tag{4.13}$$

where

$$K^{\pm}(\alpha) = (\alpha \pm k)^{\pm 1/2} K_0^{\pm}(\alpha), \quad -\infty < \alpha < +\infty,$$

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$$K_0(\alpha) = \exp\left\{\frac{\alpha}{\pi i}\int_0^\infty \log\frac{\sinh\gamma b_+ \sinh\gamma b_-}{\sinh\gamma(b_+ + b_-)}\frac{d\beta}{\beta^2 - \alpha^2}\right\}, \quad \alpha \notin (-\infty, +\infty), \qquad (4.14)$$

and $K_0^\pm(\alpha)$, $-\infty < \alpha < +\infty$, are defined by the Sokhotsky-Plemelj formulas. Then we rewrite vector equation (4.11) as

$$K^-(\alpha)\Phi_2^-(\alpha) - \Psi_-(\alpha) = -K^+(\alpha)\Phi_1^+(\alpha) - \frac{e^{i\alpha}K^+(\alpha)\Phi_2^+(\alpha)}{g(\alpha)} - \Psi_-^+(\alpha),$$

$$\frac{\Phi_2^+(\alpha)}{K^+(\alpha)} + \Psi^+(\alpha) = -\frac{\Phi_1^-(\alpha)}{K^-(\alpha)} - \frac{e^{-i\alpha}\Phi_2^-(\alpha)}{K^-(\alpha)g(\alpha)} + \Psi_+^-(\alpha), \qquad (4.15)$$

where

$$\Psi_-(\alpha) = \int_{-\infty}^\infty \frac{K^-(\beta)F^+(\beta)d\beta}{\beta - \alpha}, \quad \Psi_+(\alpha) = \int_{-\infty}^\infty \frac{F^-(\beta)d\beta}{K^+(\beta)(\beta - \alpha)}. \qquad (4.16)$$

The right-hand sides of the first and second equations in (4.15) are analytic everywhere in the domains $C^+ : \operatorname{Im}\alpha > 0$ and $C^- : \operatorname{Im}\alpha < 0$, respectively, except at the zeros of the function $g(\alpha)$. Denote by $\alpha_{m\pm}$ the zeros of the functions $\sinh\gamma b_\pm$ lying in the upper half-plane,

$$\alpha_{m\pm} = \sqrt{k^2 - \frac{\pi^2 m^2}{b_\pm^2}}, \quad \operatorname{Im}\alpha_{m\pm} > 0, \quad m = 1, 2, \ldots. \qquad (4.17)$$

Then in C^+, the function $g(\alpha)$ has simple zeros at the points $\alpha = k$ and $\alpha = \alpha_{m\pm}$, $m = 1, 2, \ldots$. In the lower half-plane, the zeros are $\alpha = -k$ and $\alpha = -\alpha_{m\pm}$, $m = 1, 2, \ldots$. In order to remove the simple poles of the right-hand sides of equations (4.15), we introduce the functions

$$\Omega^+(\alpha) = \frac{A_0^+}{\alpha + k} + \sum_{m=1}^\infty \left(\frac{A_{m+}^+}{\alpha + \alpha_{m+}} + \frac{A_{m-}^+}{\alpha + \alpha_{m-}}\right),$$

$$\Omega^-(\alpha) = \frac{A_0^-}{\alpha - k} + \sum_{m=1}^\infty \left(\frac{A_{m+}^-}{\alpha - \alpha_{m+}} + \frac{A_{m-}^-}{\alpha - \alpha_{m-}}\right). \qquad (4.18)$$

Assume that the residues of the functions $\Omega^-(\alpha)$ and $\Omega^+(\alpha)$ at their poles are chosen such that, when they are subtracted from the left and right-hand sides of the first and second equations in (4.15), respectively, the poles are removed. On employing the Liouville theorem we find the solution of the RHP

$$\Phi_2^-(\alpha) = \frac{\Psi_-(\alpha) + \Omega^-(\alpha)}{K^-(\alpha)}, \quad \Phi_2^+(\alpha) = K^+(\alpha)[-\Psi_+^+(\alpha) + \Omega^+(\alpha)],$$

$$\Phi_1^-(\alpha) = K^-(\alpha)[\Psi_+^-(\alpha) - \Omega^+(\alpha)] - \frac{e^{-i\alpha}}{K^-(\alpha)g(\alpha)}[\Psi_-^+(\alpha) + \Omega^-(\alpha)],$$

$$\Phi_1^+(\alpha) = -\frac{\Psi_-^+(\alpha) + \Omega^-(\alpha)}{K^+(\alpha)} + \frac{e^{i\alpha}K^+(\alpha)}{g(\alpha)}[\Psi_+^+(\alpha) - \Omega^+(\alpha)]. \qquad (4.19)$$

It is directly verified that $\Phi_1^+(\alpha) = e^{i\alpha}\Phi_1^-(\alpha)$. The coefficients $A_0^\pm$, $A_{m+}^\pm$ and $A_{m-}^\pm$ need to be fixed by the conditions

$$\operatorname*{res}_{\alpha = \pm k}\Phi_1^\pm(\alpha) = 0, \quad \operatorname*{res}_{\alpha = \alpha_{m\pm}}\Phi_1^+(\alpha) = 0, \quad \operatorname*{res}_{\alpha = -\alpha_{m\pm}}\Phi_1^-(\alpha) = 0, \quad m = 1, 2, \ldots. \qquad (4.20)$$

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These conditions if satisfied guarantee that the zeros of the function $g(\alpha)$ are removable singularities of the functions $\Phi_1^\pm(\alpha)$. On calculating the residues we rewrite the conditions (4.20) as a system of linear algebraic equations

$$A_0^\mp \pm \delta_0 e^{ik} \left[\frac{A_0^\pm}{2k} + \sum_{m=1}^\infty \left(\frac{A_{m+}^\pm}{k + \alpha_{m+}} + \frac{A_{m-}^\pm}{k + \alpha_{m-}} \right) \right] = \pm \delta_0 e^{ik} \Psi_\pm^\pm(\pm k),$$

$$A_{n\pm}^- + \delta_{n\pm} e^{i\alpha_{n\pm}} \left[\frac{A_0^+}{\alpha_{n\pm} + k} + \sum_{m=1}^\infty \left(\frac{A_{m+}^+}{\alpha_{n\pm} + \alpha_{m+}} + \frac{A_{m-}^+}{\alpha_{n\pm} + \alpha_{m-}} \right) \right] = \delta_{n\pm} e^{i\alpha_{n\pm}} \Psi_+^+(\alpha_{n\pm}),$$

$$A_{n\pm}^+ - \delta_{n\pm} e^{i\alpha_{n\pm}} \left[\frac{A_0^-}{\alpha_{n\pm} + k} + \sum_{m=1}^\infty \left(\frac{A_{m+}^-}{\alpha_{n\pm} + \alpha_{m+}} + \frac{A_{m-}^-}{\alpha_{n\pm} + \alpha_{m-}} \right) \right]$$

$$= -\delta_{n\pm} e^{i\alpha_{n\pm}} \Psi_-^-(-\alpha_{n\pm}), \quad n = 1, 2, \ldots. \tag{4.21}$$

Here,

$$\delta_{n\pm} = \frac{(-1)^n (1 + k/\alpha_{n\pm})[K_0^+(\alpha_{n\pm})]^2 \sin[\pi n(b_+ + b_-)/b_\pm]}{b_\pm \sin(\pi n b_\mp/b_\pm)},$$

$$\delta_0 = \left(\frac{1}{b_+} + \frac{1}{b_-} \right) [K_0^+(k)]^2. \tag{4.22}$$

If we assume the symmetry of the problem, $b_- = b_+ = b$, then the formulas can be simplified. Indeed, $g(\alpha)$ now becomes $\frac{1}{2}\gamma \tanh \gamma b$, and therefore its zeros are given by $\alpha_m = \sqrt{k^2 - \pi^2 m^2/b^2}$ ($\alpha_0 = k$), $\operatorname{Im} \alpha_m > 0$, $m = 0, 1, \ldots$. The functions $\Omega^\pm(\alpha)$ are represented by the series

$$\Omega^\pm(\alpha) = \sum_{m=0}^\infty \frac{A_m^\pm}{\alpha \pm \alpha_m}, \tag{4.23}$$

and the infinite systems reduce to

$$A_n^\pm \mp \delta_n e^{i\alpha_n} \sum_{m=0}^\infty \frac{A_m^\mp}{\alpha_n + \alpha_m} = h_n^\pm, \quad n = 0, 1, \ldots. \tag{4.24}$$

Here,

$$h_n^\pm = \mp \delta_n e^{i\alpha_n} \Psi_\mp^\mp(\mp \alpha_n), \quad \delta_n = \frac{2(\alpha_n + k)}{\alpha_n b} [K_0^+(\alpha_n)]^2. \tag{4.25}$$

Notice that in both cases, $b_- \neq b_+$ and $b_- = b_+$, the unknowns A_n^+ and A_n^- exponentially decay as $n \to \infty$, $|A_n^\pm| < c_\pm e^{qn}$, $c_\pm$ and q are some positive constants.

5   Conclusion

We have developed further the algorithm for the vector RHP when its matrix coefficient entries are meromorphic and almost periodic functions. In the simplest case, when the meromorphic functions have a finite number of poles and zeros (rational functions), regardless of the dimension of the vector RHP, the exact solution can always been constructed. We have illustrated this approach by solving the inhomogeneous heat equation with a piece-wise constant diffusivity. In the case of two discontinuities, 0 and ∞, we have derived a simple representation of the solution that generalizes for the discontinuous case the classical Poisson formula for an infinite rod.

We have also shown that, when the meromorphic functions have periodic zeros and poles only, then it is possible to derive the Wiener-Hopf factors of the RHP matrix

coefficient in a closed form in terms of the hypergeometric functions. This technique has been employed for finding the exact solution of the vector RHP associated with a mixed boundary value problem for the Laplace equation in a wedge when on finite segments of the wedge sides, the function is prescribed, and on two semi-infinite segments, the normal derivative is known, and no symmetry that would allow for decoupling of the problem is assumed.

In the general case, when the zeros or poles are not periodic, the solution can be derived by quadratures and some exponentially convergent series. The series coefficients solve an infinite system of linear algebraic equations of the second kind whose rate of converge is exponential. This technique has been used for solving the Neumann boundary value problem for a strip with a finite slit inside parallel to the strip boundaries.

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